

# Complexity Theory

## Part One

It may be that since one is customarily concerned with existence, [...] finiteness, and so forth, one is not inclined to take seriously the question of the existence of a *better-than-finite* algorithm.

- Jack Edmonds, "Paths, Trees, and Flowers"

# A Decidable Problem

- Consider the following problem:

*Given two regular expressions  $R_1$  and  $R_2$ , determine whether  $R_1$  and  $R_2$  have the same language.*

- This problem is indeed decidable.
  - We autograded your regular expressions in Problem Set Seven. The algorithm we used is 100% accurate.
- **Theorem:** There is no algorithm for solving this problem whose runtime is  $O(2^{m+n})$ , where  $m$  and  $n$  are the lengths of the input regular expressions.

# The Limits of Decidability

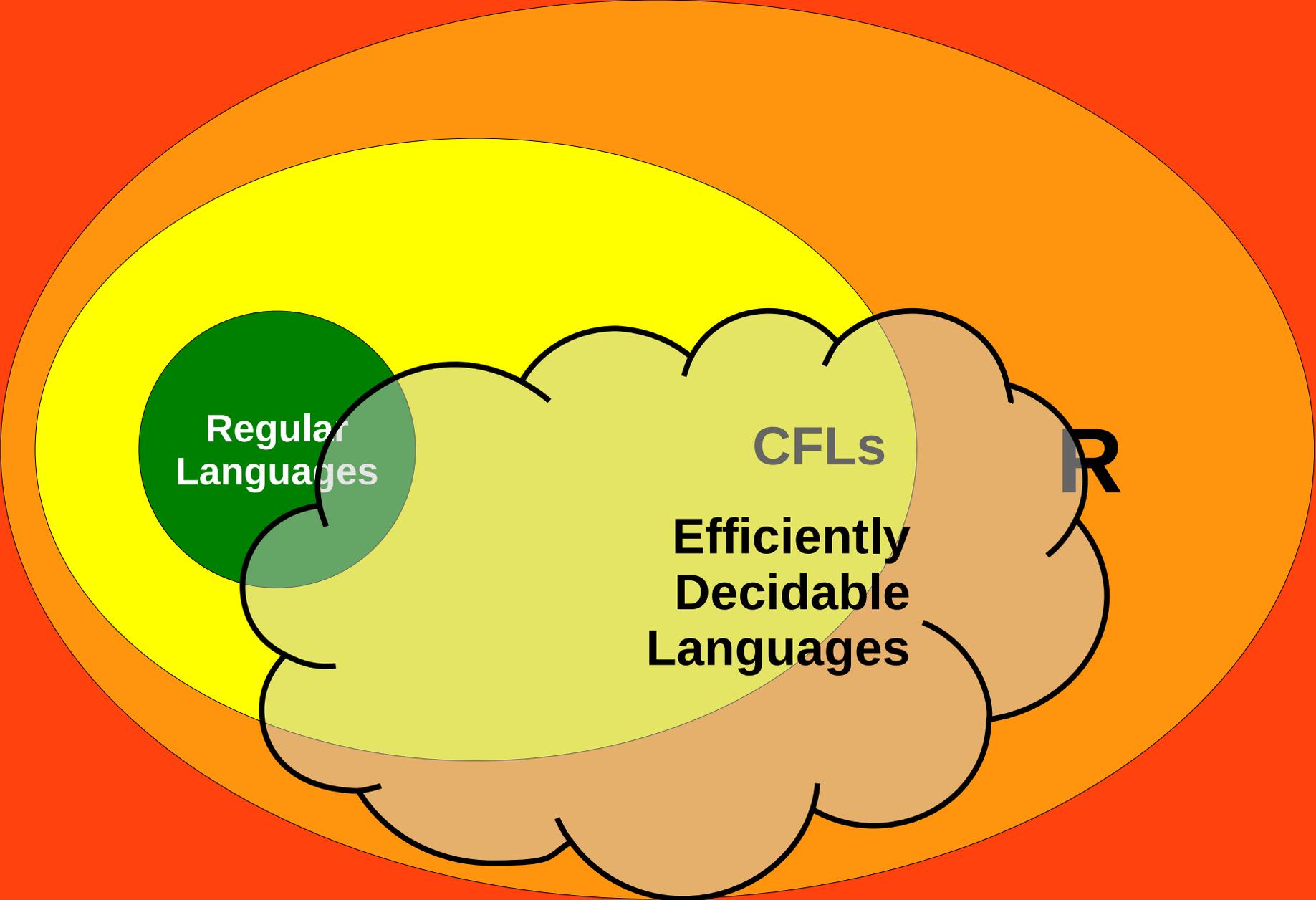
- The fact that a problem is decidable does not mean that it is *feasibly* decidable.
- In ***computability theory***, we ask the question  
What problems can be solved by a computer?
- In ***complexity theory***, we ask the question  
What problems can be solved ***efficiently*** by a computer?
- In the remainder of this course, we will explore this question in more detail.

# Where We've Been

- The class **R** represents problems that can be solved by a computer.
- The class **RE** represents problems where “yes” answers can be verified by a computer.

# Where We're Going

- The class **P** represents problems that can be solved *efficiently* by a computer.
- The class **NP** represents problems where “yes” answers can be verified *efficiently* by a computer.



Regular Languages

CFLs

Efficiently Decidable Languages

Undecidable Languages

R

# The Setup

- In order to study computability, we needed to answer these questions:
  - What is “computation?”
  - What is a “problem?”
  - What does it mean to “solve” a problem?
- To study complexity, we need to answer these questions:
  - What does “complexity” even mean?
  - What is an “efficient” solution to a problem?

# Measuring Efficiency

We have a program written in your Favorite Programming Language that's a decider for some problem.

The program is correct in the sense that it always produces the right output for any given input.

How might we measure the "efficiency" of that solution?

The number of lines of code in the program.

How deeply-nested the loops or recursion in the program are.

- **How much time it takes for the program to solve the problem.**

How much memory it takes for the program to solve the problem.

How much power it takes for

How much network communication it takes for the program to solve the problem.

...

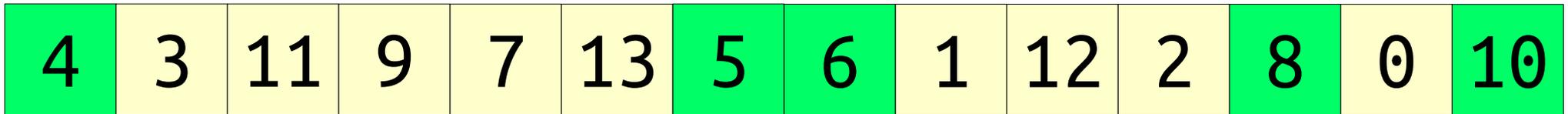
We're going to focus on this measure of "efficiency," but that doesn't mean these other ones aren't interesting! There's tons of research on them.

What is an efficient algorithm?

# Searching Finite Spaces

- Many decidable problems can be solved by searching over a large but finite space of possible options.
- Searching this space might take a staggeringly long time, but only finite time.
- From a decidability perspective, this is totally fine.
- From a complexity perspective, this may be totally unacceptable.

# A Sample Problem



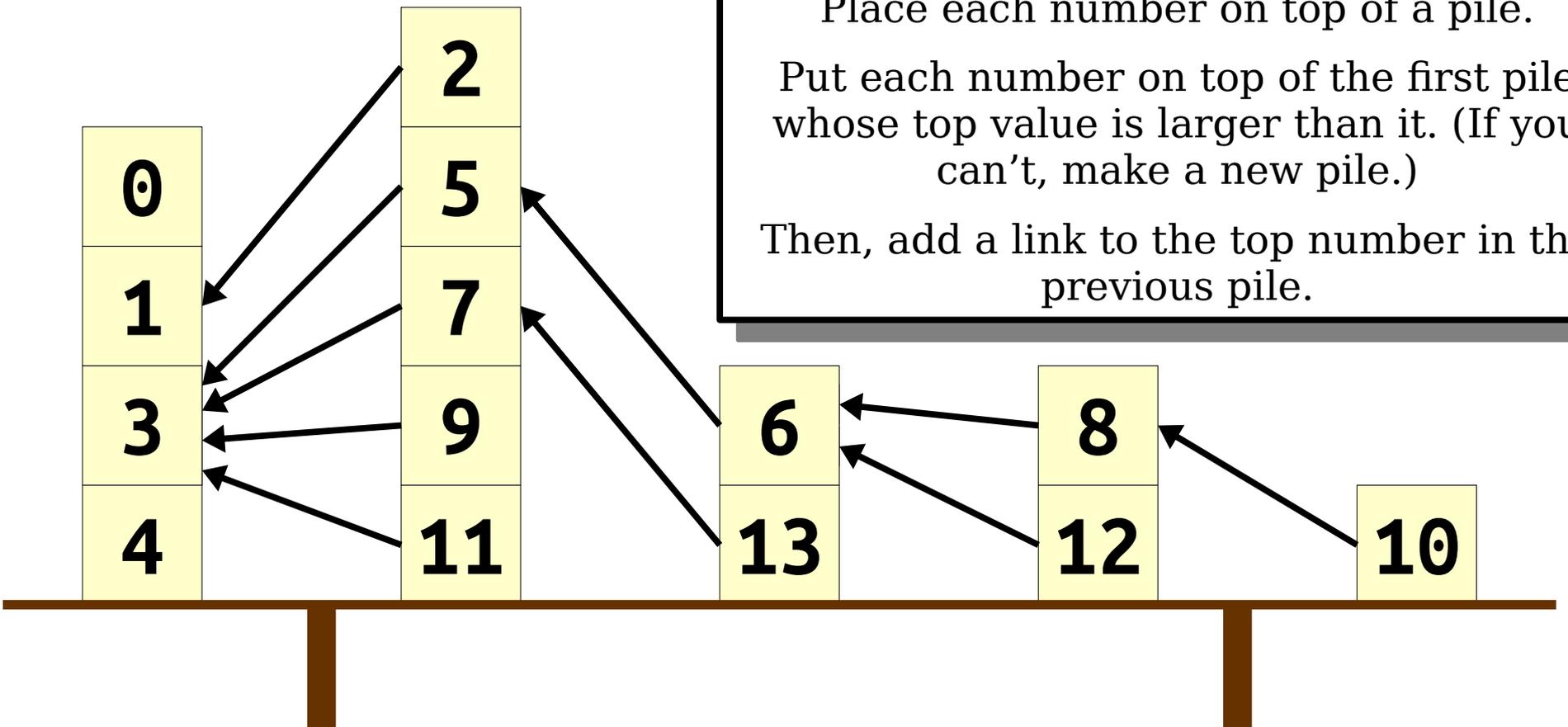
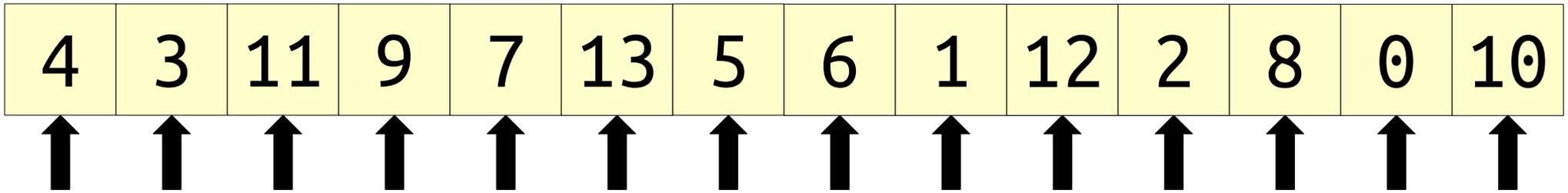
Goal: Find the length of the longest increasing subsequence of this sequence.

# Longest Increasing Subsequences

- ***One possible algorithm:*** try all subsequences, find the longest one that's increasing, and return that.
- There are  $2^n$  subsequences of an array of length  $n$ .
  - (Each subset of the elements gives back a subsequence.)
- Checking all of them to find the longest increasing subsequence will take time  $O(n \cdot 2^n)$ .
- ***Fact:*** the age of the universe is about  $4.3 \times 10^{26}$  nanoseconds. That's about  $2^{85}$  nanoseconds.
- Practically speaking, this algorithm doesn't terminate if you give it an input of size 100 or more.

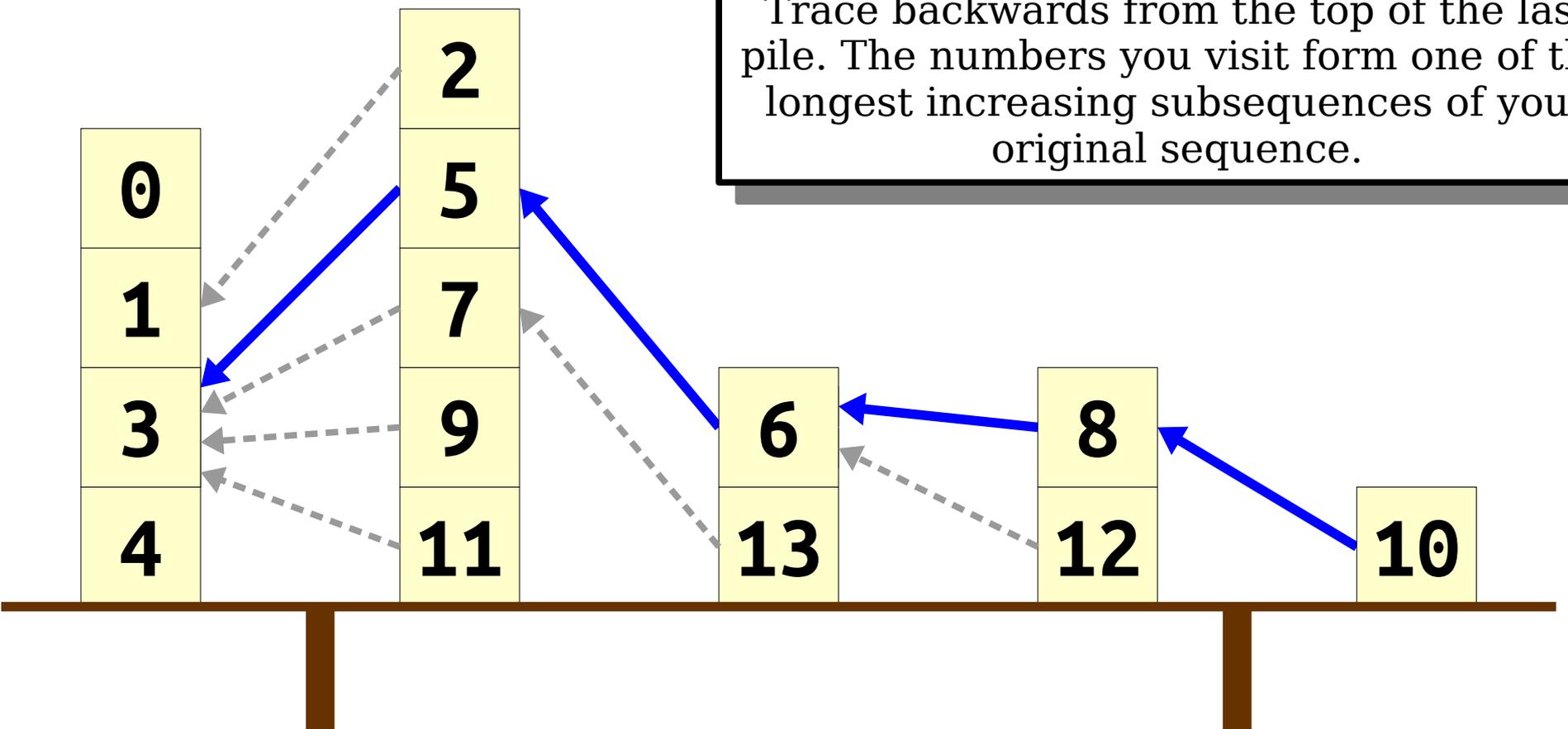
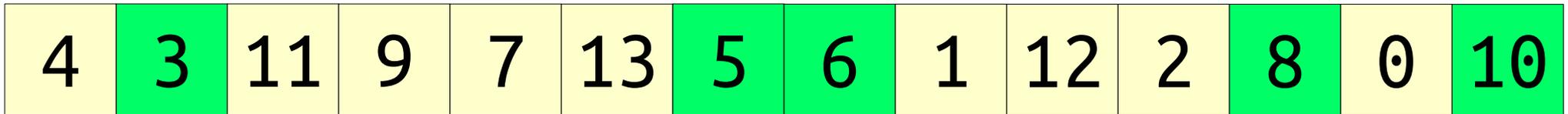
# A Different Approach

# Patience Sorting



Place each number on top of a pile.  
Put each number on top of the first pile whose top value is larger than it. (If you can't, make a new pile.)  
Then, add a link to the top number in the previous pile.

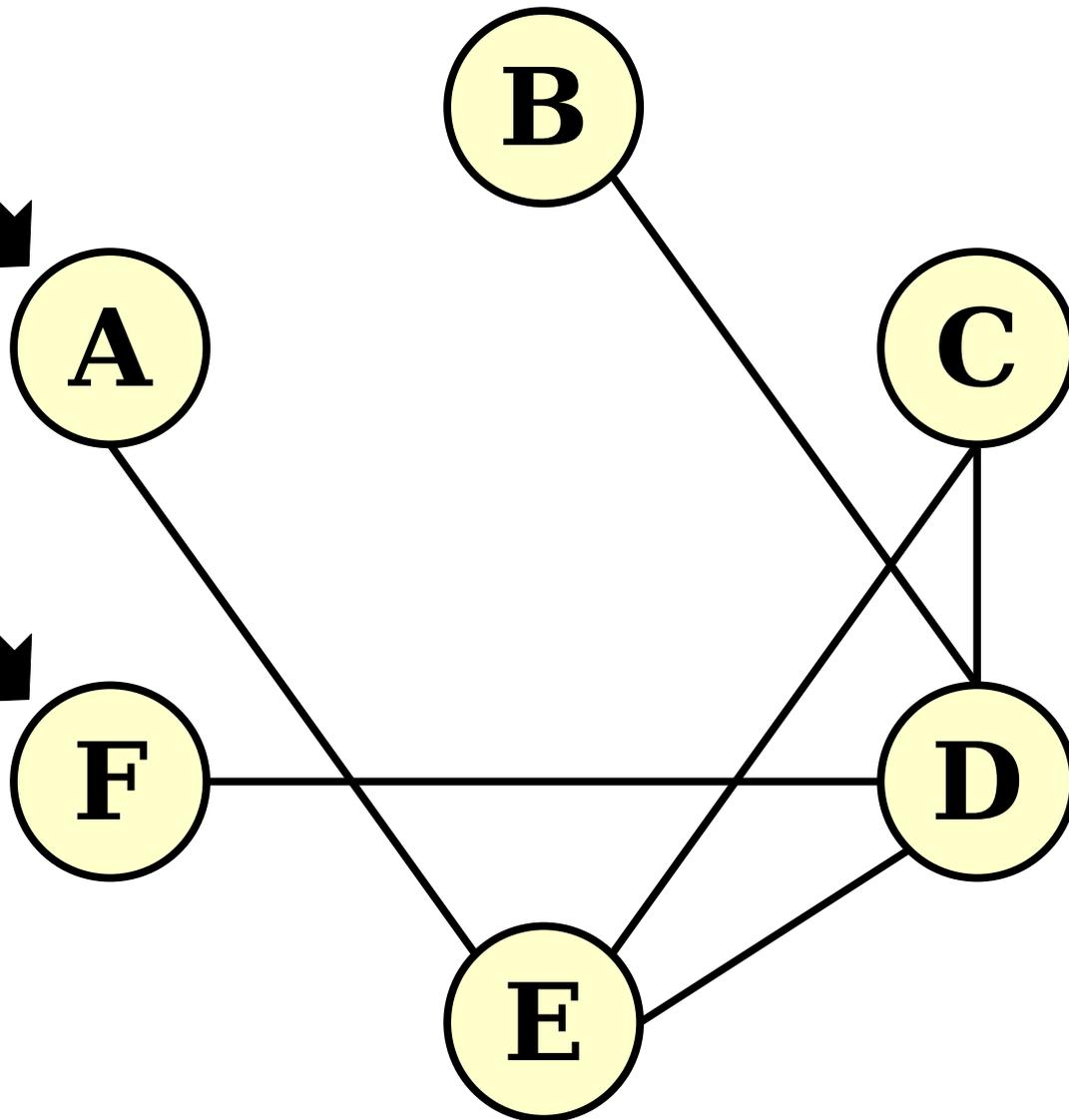
# Patience Sorting



# Longest Increasing Subsequences

- **Theorem:** There is an algorithm that can find the longest increasing subsequence of an array in time  $O(n^2)$ .
  - It's the previous **patience sorting** algorithm, with some clever implementation tricks.
- This algorithm works by exploiting particular aspects of how longest increasing subsequences are constructed. It's not immediately obvious that it works correctly.
- **CS161-Style Exercise 1:** Prove that this procedure always works!
- **CS161-Style Exercise 2:** Show that you can implement this algorithm in time  $O(n \log n)$ .

# Another Problem



Goal: Determine the length of the shortest path from **F** to **A** in this graph.

# Shortest Paths

- It is possible to find the shortest path in a graph by listing off all sequences of nodes in the graph in ascending order of length and finding the first that's a path.
- This takes time  $O(n \cdot n!)$  in an  $n$ -node graph.
- For reference:  $29!$  nanoseconds is longer than the lifetime of the universe.

# Shortest Paths

- ***Theorem:*** It's possible to find the shortest path between two nodes in an  $n$ -node,  $m$ -edge graph in time  $O(m + n)$ .
- ***Proof idea:*** Use breadth-first search!
- The algorithm is a bit nuanced. It uses some specific properties of shortest paths and the proof of correctness is nontrivial.

# For Comparison

- ***Longest increasing subsequence:***
  - Naive:  $O(n \cdot 2^n)$
  - Fast:  $O(n^2)$
- ***Shortest path problem:***
  - Naive:  $O(n \cdot n!)$
  - Fast:  $O(n + m)$ .

# Defining Efficiency

- When dealing with problems that search for the “best” object of some sort, there are often at least exponentially many possible options.
- Brute-force solutions tend to take at least exponential time to complete.
- Clever algorithms often run in time  $O(n)$ , or  $O(n^2)$ , or  $O(n^3)$ , etc.

# Polynomials and Exponentials

- An algorithm runs in ***polynomial time*** if its runtime is some polynomial in  $n$ .
  - That is, time  $O(n^k)$  for some constant  $k$ .
- Polynomial functions “scale well.”
  - Small changes to the size of the input do not typically induce enormous changes to the overall runtime.
- Exponential functions scale terribly.
  - Small changes to the size of the input induce huge changes in the overall runtime.

# The Cobham-Edmonds Thesis

A language  $L$  can be ***decided efficiently*** if there is a TM that decides it in polynomial time.

Equivalently,  $L$  can be decided efficiently if it can be decided in time  $O(n^k)$  for some  $k \in \mathbb{N}$ .

Like the Church-Turing thesis, this is ***not*** a theorem!

It's an assumption about the nature of efficient computation, and it is somewhat controversial.

# The Cobham-Edmonds Thesis

- Efficient runtimes:
  - $4n + 13$
  - $n^3 - 2n^2 + 4n$
  - $n \log \log n$
- “Efficient” runtimes:
  - $n^{1,000,000,000,000}$
  - $10^{500}$
- Inefficient runtimes:
  - $2^n$
  - $n!$
  - $n^n$
- “Inefficient” runtimes:
  - $n^{0.0001 \log n}$
  - $1.0000000001^n$

# Why Polynomials?

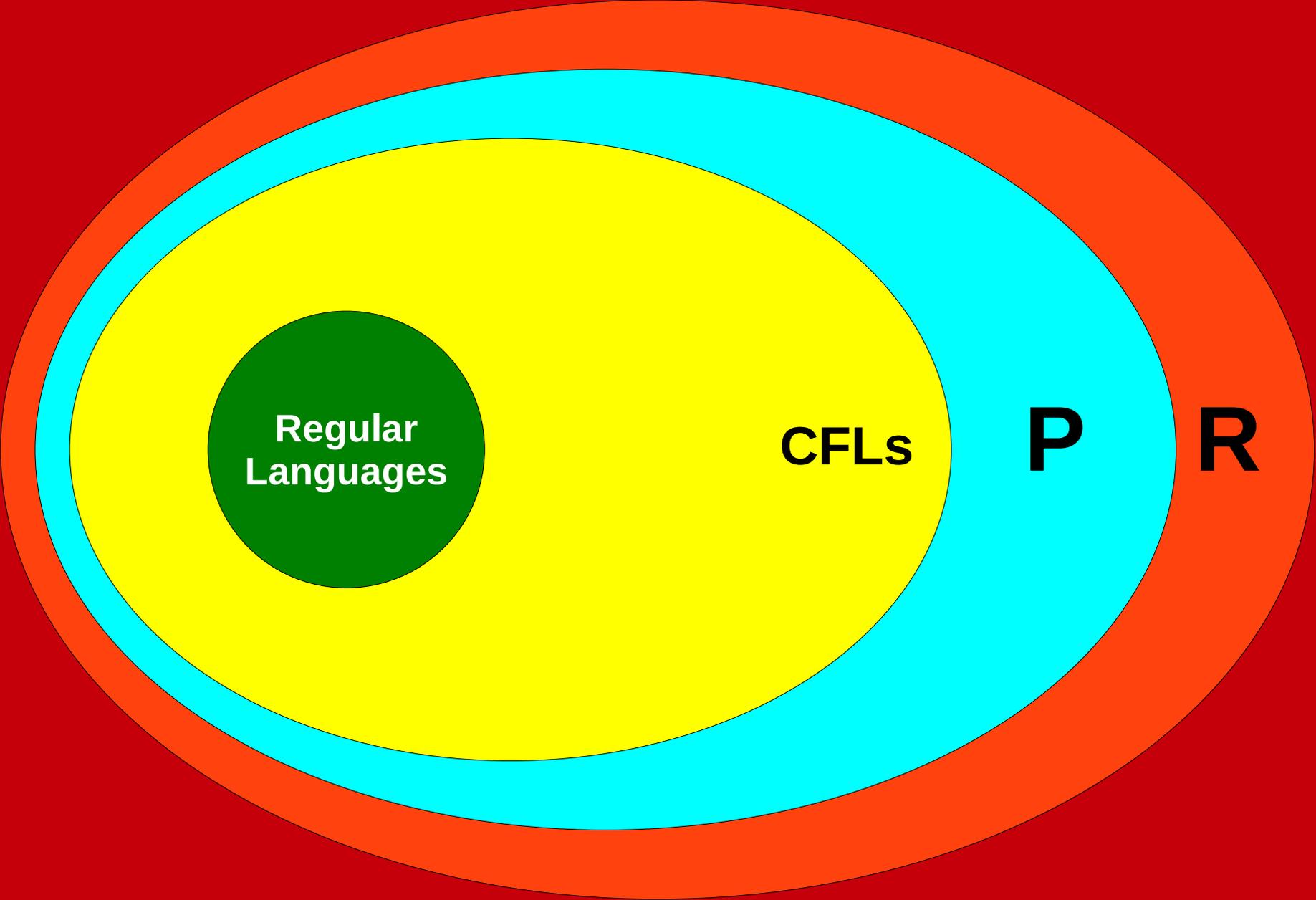
- Polynomial time *somewhat* captures efficient computation, but has a few edge cases.
- However, polynomials have very nice mathematical properties:
  - The sum of two polynomials is a polynomial. (Running one efficient algorithm, then another, gives an efficient algorithm.)
  - The product of two polynomials is a polynomial. (Running one efficient algorithm a “reasonable” number of times gives an efficient algorithm.)
  - The *composition* of two polynomials is a polynomial. (Using the output of one efficient algorithm as the input to another efficient algorithm gives an efficient algorithm.)

# The Complexity Class **P**

- The ***complexity class P*** (for ***p***olynomial time) contains all problems that can be solved in polynomial time.
- Formally:  
$$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$$
- Assuming the Cobham-Edmonds thesis, a language is in **P** if it can be decided efficiently.

# Examples of Problems in **P**

- All regular languages are in **P**.
  - All have linear-time TMs.
- All CFLs are in **P**.
  - Requires a more nuanced argument (the *CYK algorithm* or *Earley's algorithm*.)
- And a *ton* of other problems are in **P** as well.
  - Curious? Take CS161!



**Regular  
Languages**

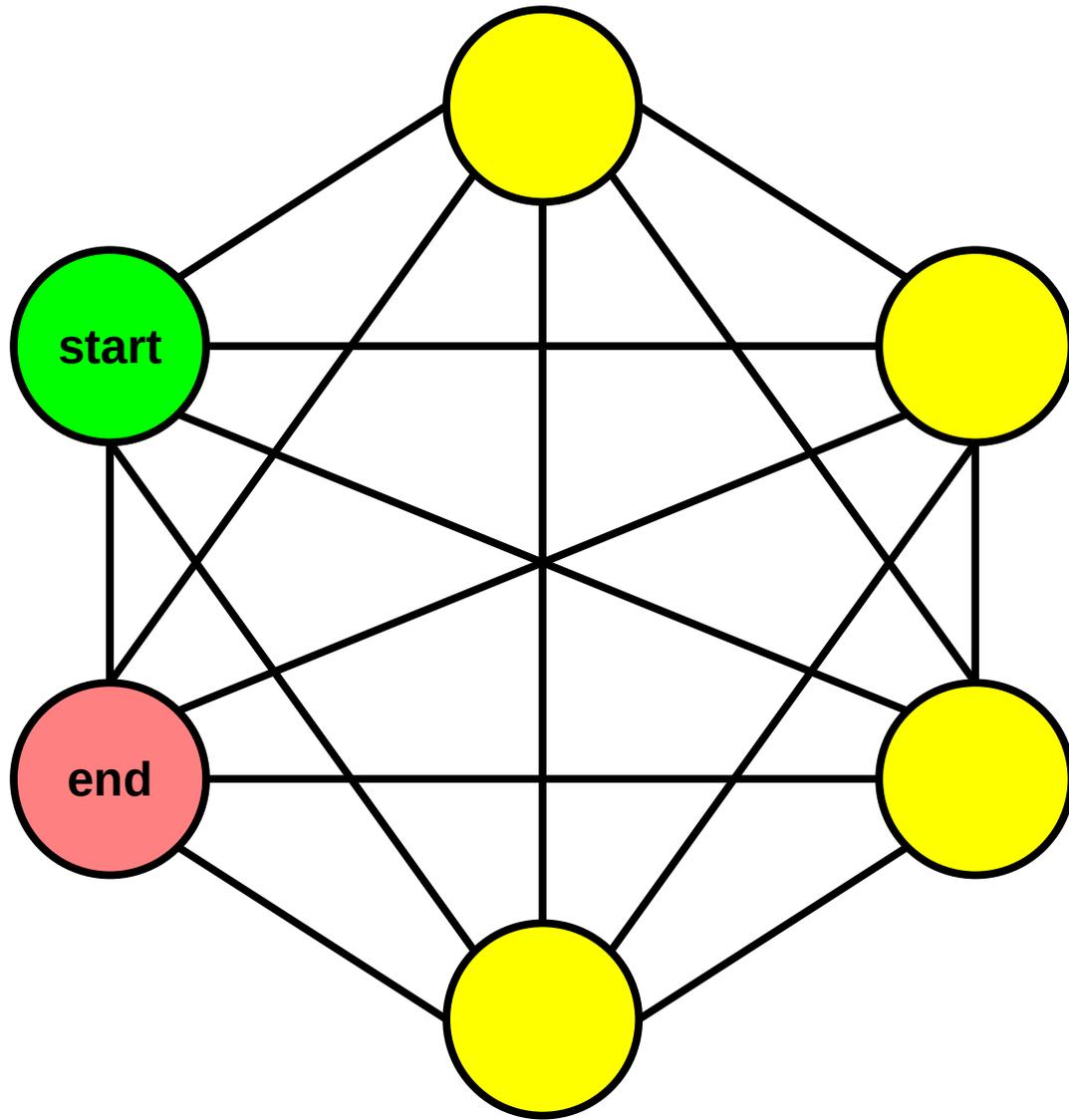
**CFLs**

**P**

**R**

**Undecidable Languages**

What *can't* you do in polynomial time?



How many paths  
are there from  
the start node  
to the end  
node?



How many  
subsets of this  
set are there?

# An Interesting Observation

- There are (at least) exponentially many objects of each of the preceding types.
- However, each of those objects is not very large.
  - Each simple path has length no longer than the number of nodes in the graph.
  - Each subset of a set has no more elements than the original set.
- This brings us to our next topic...

What if you need to search a large space for a single object?

# Verifiers - Again

2	5	7	9	6	4	1	8	3
4	9	1	8	7	3	6	5	2
3	8	6	1	2	5	9	4	7
6	4	5	7	3	2	8	1	9
7	1	9	5	4	8	3	2	6
8	3	2	6	1	9	5	7	4
1	6	3	2	5	7	4	9	8
5	7	8	4	9	6	2	3	1
9	2	4	3	8	1	7	6	5

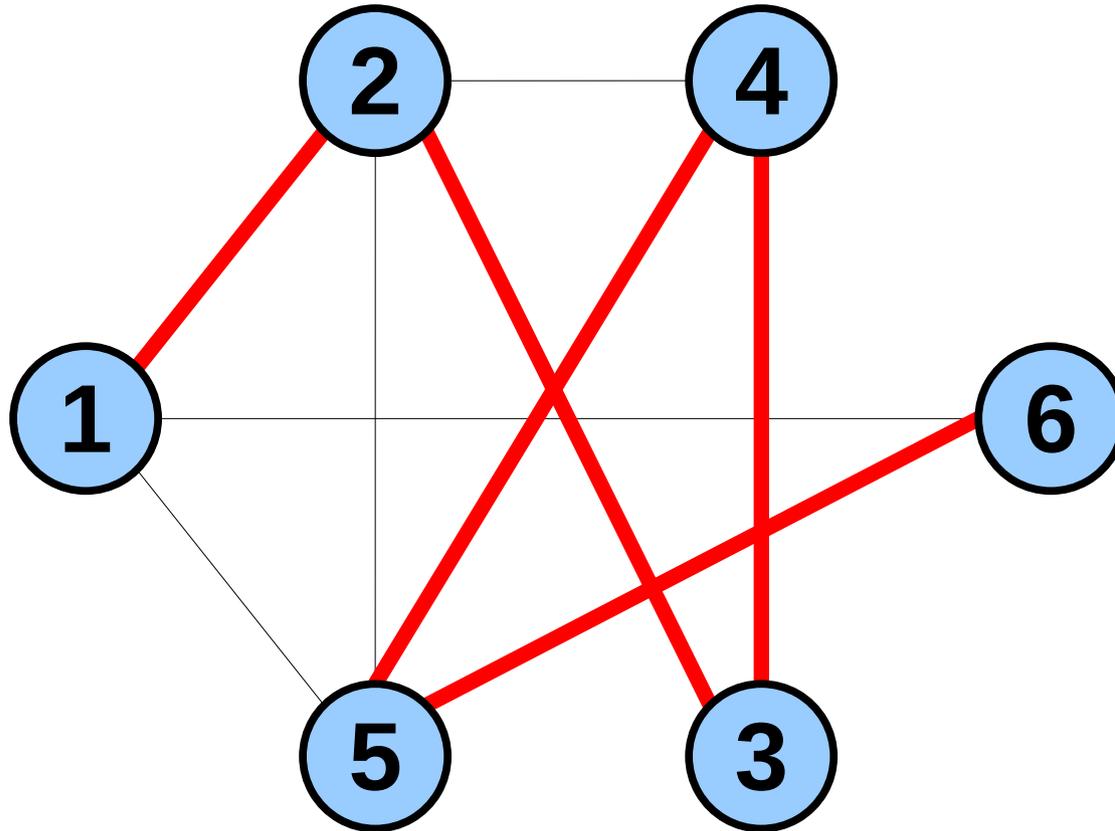
Does this Sudoku problem  
have a solution?

# Verifiers - Again

4	3	11	9	7	13	5	6	1	12	2	8	0	10
---	---	----	---	---	----	---	---	---	----	---	---	---	----

Is there an ascending subsequence of length at least 5?

# Verifiers - Again



Is there a path that goes through every node exactly once?

# Polynomial-Time Verifiers

- A ***polynomial-time verifier*** for  $L$  is a TM  $V$  such that
  - $V$  halts on all inputs.
  - $w \in L \iff \exists c \in \Sigma^*. V$  accepts  $\langle w, c \rangle$ .
  - $V$  runs “efficiently” (its runtime is  $O(|w|^k)$  for some  $k \in \mathbb{N}$ ).
  - All strings in  $L$  have “short” certificates (their lengths are  $O(|w|^r)$  for some  $r \in \mathbb{N}$ ).

# The Complexity Class **NP**

- The complexity class **NP** (*nondeterministic polynomial time*) contains all problems that can be verified in polynomial time.
- Formally:
$$\mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$$
- The name **NP** comes from another way of characterizing **NP**. If you introduce *nondeterministic Turing machines* and appropriately define “polynomial time,” then **NP** is the set of problems that an NTM can solve in polynomial time.
- **Useful fact:**  $\mathbf{NP} \subsetneq \mathbf{R}$ . Come talk to me after class if you’re curious why!

**P** = {  $L$  | there is a polynomial-time decider for  $L$  }

**NP** = {  $L$  | there is a polynomial-time verifier for  $L$  }

**R** = {  $L$  | there is a ~~polynomial-time~~ decider for  $L$  }

**RE** = {  $L$  | there is a ~~polynomial-time~~ verifier for  $L$  }

We know that  $\mathbf{R} \neq \mathbf{RE}$ .

So does that mean  $\mathbf{P} \neq \mathbf{NP}$ ?

**Time-Out for Announcements!**

***Please evaluate this course in Axess.***  
Your comments really make a difference.

# Christmas Trees Lecture

- Don Knuth, professor emeritus, will be giving a talk this Wednesday, December 7 as part of his “Christmas Trees” series.
- It’ll be in Nvidia Auditorium at 5:00PM.
- Highly recommended – Don is a legend in the field and frequently has amazing insights to share.
- (Here’s a [\*New York Times profile\*](#).)

# Problem Sets

- Problem Set 8 solutions are now up on the course website.
  - Your TAs are working on grading them, and we'll have them ready by Wednesday evening.
- Problem Set 9 is due this Friday at 2:30PM.
  - As always, come talk to us if you have any questions!

# Final Exam Logistics

- Our final exam is on ***Tuesday, December 13<sup>th</sup>*** from ***8:30AM - 11:30AM***. It'll be held in ***Hewlett 200***.
- The final exam is cumulative and covers topics from PS1 - PS9 and L00 - L27. The format is similar to that of the midterm, with a mix of short-answer questions and formal written proofs.
- Like the midterms, it's closed-book, closed-computer, and limited-note. You can bring one double-sided 8.5" × 11" notes sheet with you.
- Students with OAE accommodations: we will be reaching out later this evening to coordinate alternate final exam times.

# Preparing for the Final Exam

- We've posted a gigantic compendium of CS103 practice problems on the course website.
- You can search for problems based on the topics they cover, whether solutions are available, whether they're ones we particularly like, whether they were used on past exams, and whether they have solutions.
- As always, ***keep the TAs in the loop!*** Ask us questions if you have them, feel free to stop by office hours to discuss solutions, etc.

Your Questions

# “Advice on how best to take advantage of cumulative practice problems? (There are so many!)”

1. Identify problems that are likely to help teach you new skills. If you're having trouble on problems that involve working formal definitions, focus on those. If you've struggled more when working with the pigeonhole principle, start there.
2. Work through the problems slowly. Write out two columns, one for what you're assuming and one for what you're proving. Make a list of strategies you've seen and go through them one at a time looking for something that works. And keep detailed notes on what you've attempted that didn't work – those are helpful for later.
3. Talk to the TAs. Do this both if you eventually solved the problem and if you got stuck and couldn't figure things out. As you do, go over your notes of what you tried that worked and what you tried that didn't. Your goal is to get help debugging your thought process.

“I came to realize that I really enjoyed the content in 103. Just curious if you have any recommended classes to learn more about computational theories, and what your thoughts are on the theory track of CS majors.”

The theory track is a great option if you like this material and want a smaller community within CS. The faculty are wonderful, approachable, and super interesting, and the stuff they do feels like science fiction!

I'll give a rundown of a bunch of possible follow-up classes to CS103 in our last lecture. In the meantime, the “sequel” course to CS103 is CS154, which goes into way more detail about automata and computability than we do and then runs wild with complexity theory.

Back to CS103!

And now...

The

***Biggest Question***

in

***Theoretical Computer Science***

**P**  $\stackrel{?}{=}$  **NP**

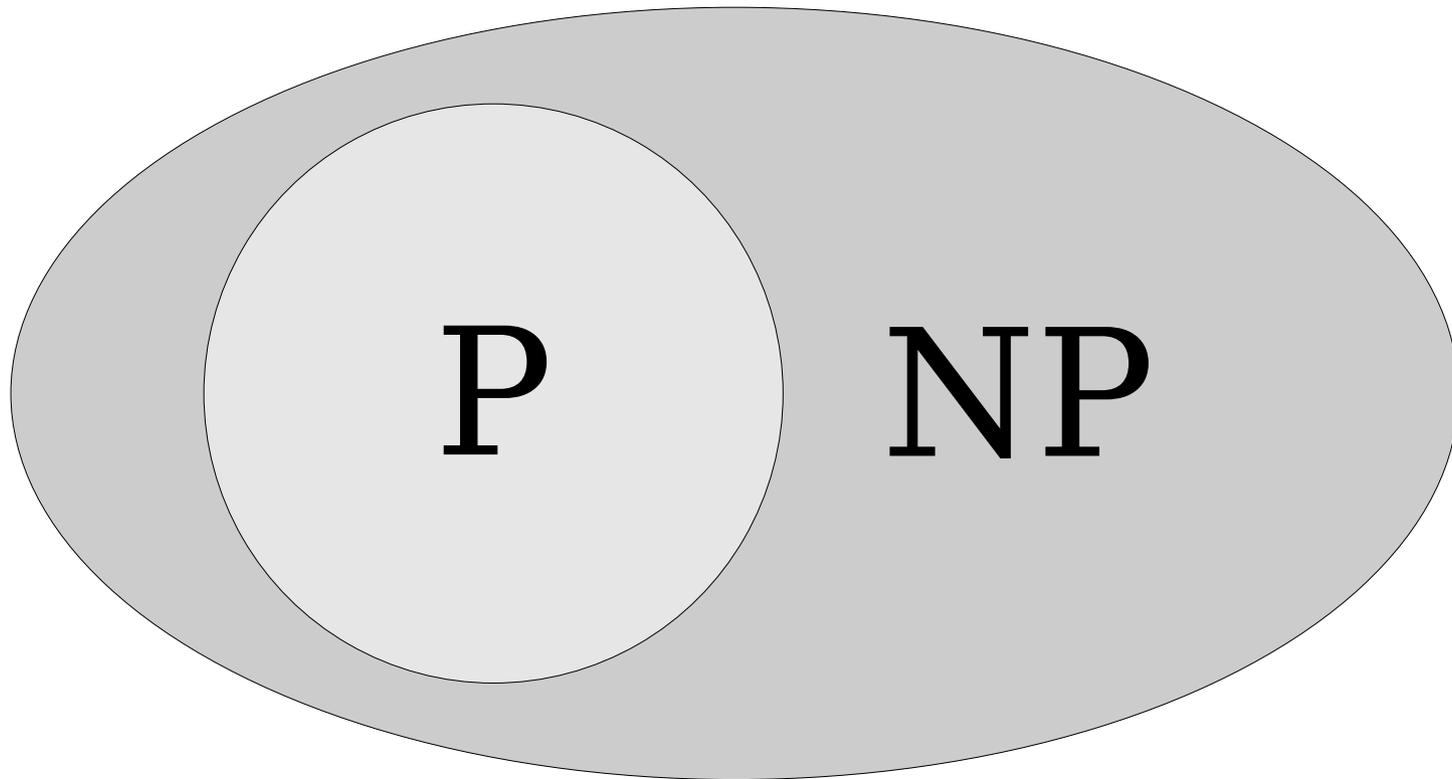
$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$

$\mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$

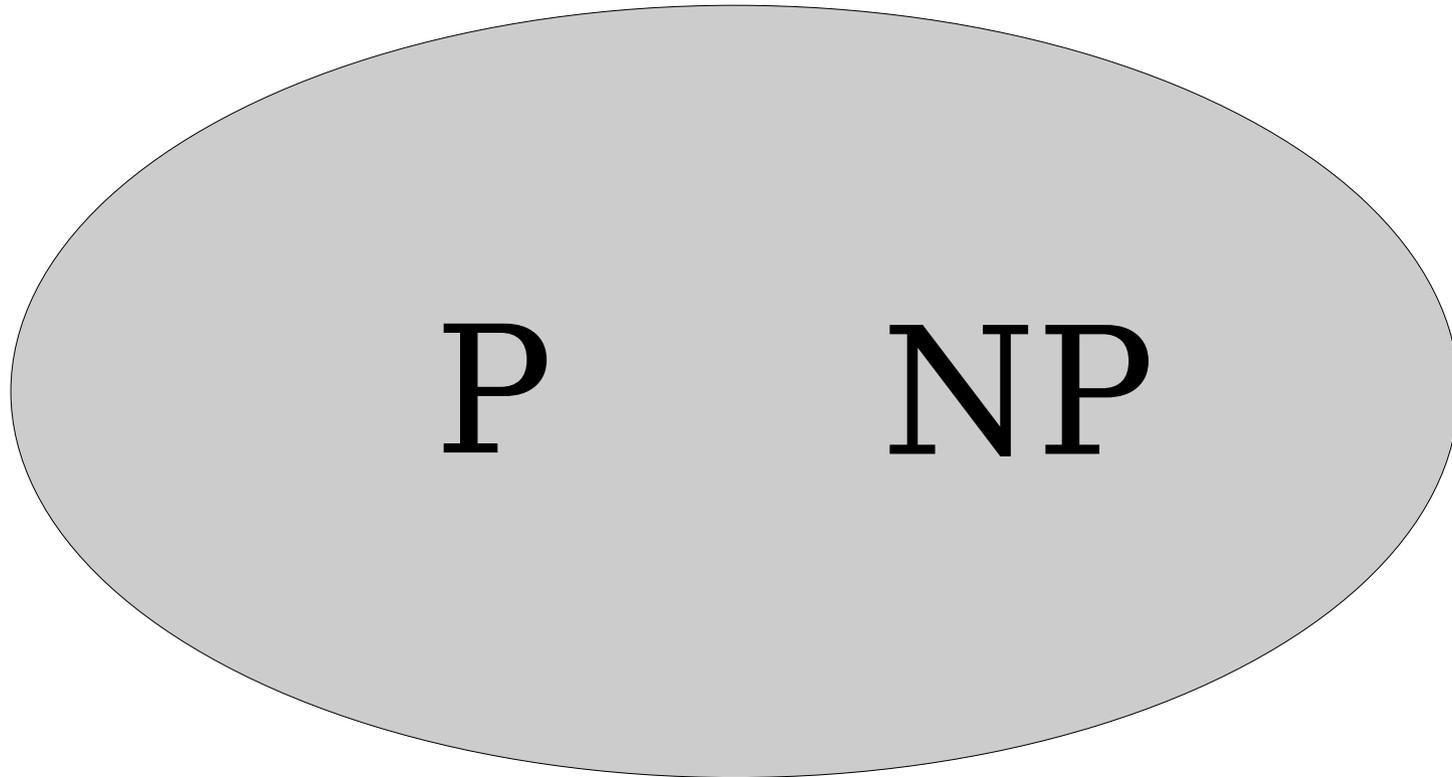


$\mathbf{P} \subseteq \mathbf{NP}$

# Which Picture is Correct?



# Which Picture is Correct?



# $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$

- The  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$  question is the most important question in theoretical computer science.
- With the verifier definition of  $\mathbf{NP}$ , one way of phrasing this question is

*If a solution to a problem can be **checked** efficiently,  
can that problem be **solved** efficiently?*

- An answer either way will give fundamental insights into the nature of computation.

# Why This Matters

- The following problems are known to be efficiently verifiable, but have no known efficient solutions:
  - Determining whether an electrical grid can be built to link up some number of houses for some price (Steiner tree problem).
  - Determining whether a simple DNA strand exists that multiple gene sequences could be a part of (shortest common supersequence).
  - Determining the best way to assign hardware resources in a compiler (optimal register allocation).
  - Determining the best way to distribute tasks to multiple workers to minimize completion time (job scheduling).
  - *And many more.*
- If  $P = NP$ , *all* of these problems have efficient solutions.
- If  $P \neq NP$ , *none* of these problems have efficient solutions.

# Why This Matters

- If **P = NP**:
  - A huge number of seemingly difficult problems could be solved efficiently.
  - Our capacity to solve many problems will scale well with the size of the problems we want to solve.
- If **P  $\neq$  NP**:
  - Enormous computational power would be required to solve many seemingly easy tasks.
  - Our capacity to solve problems will fail to keep up with our curiosity.

# What We Know

- Resolving  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$  has proven *extremely difficult*.
- In the past 49 years:
  - Not a single correct proof either way has been found.
  - Many types of proofs have been shown to be insufficiently powerful to determine whether  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ .
  - A majority of computer scientists believe  $\mathbf{P} \neq \mathbf{NP}$ , but this isn't a large majority.
- Interesting read: Interviews with leading thinkers about  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ :
  - <https://www.cs.umd.edu/~gasarch/papers/poll.pdf>

# The Million-Dollar Question

**CHALLENGE ACCEPTED**



The Clay Mathematics Institute has offered a ***\$1,000,000 prize*** to anyone who proves or disproves  **$P = NP$** .

“My hunch is that [**P**  $\stackrel{?}{=}$  **NP**] will be solved by a young researcher who is not encumbered by too much conventional wisdom about how to attack the problem.”

– Prof. Richard Karp

*(The guy who first popularized the P  $\stackrel{?}{=}$  NP problem.)*

What do we know about  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ ?

# Adapting our Techniques

**P** = {  $L$  | there is a polynomial-time decider for  $L$  }

**NP** = {  $L$  | there is a polynomial-time verifier for  $L$  }

**R** = {  $L$  | there is a ~~polynomial-time~~ decider for  $L$  }

**RE** = {  $L$  | there is a ~~polynomial-time~~ verifier for  $L$  }

We know that  $\mathbf{R} \neq \mathbf{RE}$ .

So does that mean  $\mathbf{P} \neq \mathbf{NP}$ ?

# A Problem

- The **R** and **RE** languages correspond to problems that can be decided and verified, *period*, without any time bounds.
- To reason about what's in **R** and what's in **RE**, we used two key techniques:
  - **Universality**: TMs can simulate other TMs.
  - **Self-Reference**: TMs can get their own source code.
- Why can't we just do that for **P** and **NP**?

***Theorem (Baker-Gill-Solovay):*** Any proof that purely relies on universality and self-reference cannot resolve  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ .

***Proof:*** Take CS154!

So how *are* we going to  
reason about **P** and **NP**?

# Next Time

- ***Reducibility***
  - A technique for connecting problems to one another.
- ***NP-Completeness***
  - What are the hardest problems in **NP**?